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Journal of Geometry and Physics 28 (1998) 85–93
JOURNAL OF GEOMETRY AND PHYSICS

Constant mean curvature spacelike hypersurfaces with spherical boundary in the Lorentz–Minkowski space

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Received 11 February 1998

Abstract

We study compact spacelike hypersurfaces (necessarily with non-empty boundary) with constant mean curvature in the $(n + 1)$-dimensional Lorentz–Minkowski space. In particular, when the boundary is a round sphere we prove that the only such hypersurfaces are the hyperplanar round balls (with zero mean curvature) and the hyperbolic caps (with non-zero constant mean curvature).

Subj. Class.: Differential geometry, General relativity
1991 MSC: Primary 53C42; Secondary 53B30 and 53C50
Keywords: Spacelike hypersurfaces; Constant mean curvature; Flux formula

1. Introduction

From a physical point of view, the interest of constant mean curvature spacelike hypersurfaces in Lorentzian spaces is motivated by their role in the study of different problems in general relativity. Actually, in his classical paper [9], Lichnerowicz showed that the Cauchy problem of the Einstein equation with initial conditions on a maximal (zero mean curvature spacelike) hypersurface has a particularly nice form, reducing to a linear differential system of first order and to a non-linear second order elliptic differential equation. As for spacelike hypersurfaces with non-zero constant mean curvature, they are convenient for studying the propagation of gravitational waves [10,15]. We also refer the reader to the survey papers [7,10], and references therein for other reasons justifying their importance in relativity.

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PII: S0393-0440(98)00014-X

From a mathematical point of view, their interest is also motivated by the fact that these hypersurfaces exhibit nice Bernstein-type properties. Actually, the Bernstein problem for maximal hypersurfaces in the Lorentz–Minkowski space $\mathbf{L}^{n+1}$ was first studied by Calabi [5] (for $n \leq 4$), and Cheng and Yau [6] (for arbitrary n), who proved that the only complete maximal hypersurfaces in $\mathbf{L}^{n+1}$ are the spacelike hyperplanes. In the case of the de Sitter space $\mathbf{S}_1^{n+1}$, Akutagawa [1] showed that if a complete spacelike hypersurface has constant mean curvature H satisfying $0 \leq H^2 \leq 1$ when $n = 2$, or $0 \leq H^2 < 4(n-1)/n^2$ when $n \geq 3$, then the hypersurface must be totally umbilical. Related to this result, Ramanathan [13] (for $n = 2$) and Montiel [11] (for arbitrary n) showed that the only compact spacelike hypersurfaces with constant mean curvature are the totally umbilical ones.

In this paper we study compact spacelike hypersurfaces (necessarily with non-empty boundary) with constant mean curvature in $\mathbf{L}^{n+1}$. The case $n = 2$ was previously considered by the authors, jointly with López, in [2], where it was shown that the only such surfaces spanning a circle are the planar discs and the hyperbolic caps. Here we generalize that work to the n-dimensional case, obtaining the following uniqueness result.

Theorem. *The only immersed compact spacelike hypersurfaces in $\mathbf{L}^{n+1}$ with constant mean curvature H and spherical boundary are the hyperplanar balls ($H = 0$) and the hyperbolic caps ($H \neq 0$).*

Our approach is based on two general integral formulas: a flux formula (Lemma 1) and an integral inequality (Proposition 3). The two-dimensional version of these integral formulas was obtained in [2], using in an essential way the facts that the surface M carried a complex structure and its boundary ∂M was a curve. In this paper, by means of an alternative proof which works for any dimension, we extend these formulas to the general n-dimensional case. We also derive some consequences for the case where the boundary is an embedded submanifold contained in a hyperplane of $\mathbf{L}^{n+1}$.

2. Preliminaries

Let $\mathbf{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbf{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle\ ,\ \rangle = (\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2 - (\mathrm{d}x_{n+1})^2,$$

where $(x_1, \ldots, x_{n+1})$ are the canonical coordinates in $\mathbf{R}^{n+1}$. A smooth immersion $x : M \longrightarrow \mathbf{L}^{n+1}$ of an n-dimensional connected manifold M is said to be a *spacelike hypersurface* if the induced metric via x is a Riemannian metric on M, which, as usual, is also denoted by $\langle\ ,\ \rangle$. Observe that $(0, \ldots, 0, 1)$ is a unit timelike vector field globally defined on $\mathbf{L}^{n+1}$, which determines a time-orientation on $\mathbf{L}^{n+1}$. Thus we can choose a unique unit normal vector field N on M which is a future-directed timelike vector in $\mathbf{L}^{n+1}$, and hence we may assume that M is oriented by N.

In order to set up the notation to be used later, let us denote by ∇^{o} and ∇ the Levi-Civita connections of $\mathbf{L}^{n+1}$ and M, respectively. Then the Gauss and Weingarten formulas for M in $\mathbf{L}^{n+1}$ are written, respectively, as

$$\nabla^{\text{o}}_X Y = \nabla_X Y - \langle AX, Y\rangle N, \tag{1}$$

and

$$A(X) = -\nabla^{\text{o}}_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, where A stands for the shape operator associated to N.

Throughout this paper we will deal with *compact* spacelike hypersurfaces immersed in $\mathbf{L}^{n+1}$. Let us remark that there exists no closed spacelike hypersurface in $\mathbf{L}^{n+1}$. To see this, let $a \in \mathbf{L}^{n+1}$ be a fixed arbitrary vector, and consider the height function $\langle a, x\rangle$ defined on the spacelike hypersurface M. The gradient of $\langle a, x\rangle$ is

$$\nabla\langle a, x\rangle = a^{\text{T}} = a + \langle a, N\rangle N, \tag{3}$$

where $a^{\text{T}} \in \mathcal{X}(M)$ is tangent to M, so that

$$|\nabla\langle a, x\rangle|^2 = \langle a, a\rangle + \langle a, N\rangle^2 \geq \langle a, a\rangle.$$

In particular, when a is spacelike the height function has no critical points in M, so that M cannot be closed. Therefore, every compact spacelike hypersurface M necessarily has non-empty boundary ∂M. As usual, if Σ is an $(n-1)$-dimensional closed submanifold in $\mathbf{L}^{n+1}$, a spacelike hypersurface $x : M \longrightarrow \mathbf{L}^{n+1}$ is said to be a hypersurface *with boundary* Σ if the restriction of the immersion x to the boundary ∂M is a diffeomorphism onto Σ.

3. A flux formula

In what follows, $x : M \longrightarrow \mathbf{L}^{n+1}$ will be a compact spacelike hypersurface with boundary ∂M, and we will consider M oriented by a unit future-directed timelike normal vector field N. The orientation of M induces a natural orientation on ∂M as follows: a basis $\{v_1, \ldots, v_{n-1}\}$ for $T_p(\partial M)$ is positively oriented if and only if $\{u, v_1, \ldots, v_{n-1}\}$ is a positively oriented basis for T_pM, whenever $u \in T_pM$ is outward pointing. We will denote by ν the outward pointing unit conormal vector along ∂M.

For a fixed arbitrary vector $a \in \mathbf{L}^{n+1}$, let us consider the height function $\langle a, x\rangle$ defined on M. From the expression for the gradient of $\langle a, x\rangle$ in (3), and using (1) and (2), we see that the Hessian of $\langle a, x\rangle$ is given by

$$\nabla^2\langle a, x\rangle(X, Y) = \langle\nabla_X a^{\text{T}}, Y\rangle = -\langle a, N\rangle\langle AX, Y\rangle$$

for $X, Y \in \mathcal{X}(M)$. Therefore, the Laplacian of $\langle a, x\rangle$ is

$$\Delta\langle a, x\rangle = -\langle a, N\rangle \mathrm{tr}(A) = nH\langle a, N\rangle, \tag{4}$$

where $H = -(1/n)\mathrm{tr}(A)$ defines the mean curvature function of M. Integrating now (4) on M we have by the divergence theorem that

$$\oint_{\partial M} \langle \nu, a\rangle \,\mathrm{d}S = n \int_M H\langle N, a\rangle \,\mathrm{d}V. \tag{5}$$

Here $\mathrm{d}V$ stands for the n-dimensional volume element of M with respect to the induced metric and the chosen orientation, and $\mathrm{d}S$ is the induced $(n-1)$-dimensional area element on ∂M.

On the other hand, let us consider the differential $(n-1)$-form Θ_a defined on M by

$$\Theta_a(X_1, \ldots, X_{n-1}) = \det(x, X_1, \ldots, X_{n-1}, a)$$

for $X_1, \ldots, X_{n-1} \in \mathcal{X}(M)$, where det stands for the determinant in $\mathbf{R}^{n+1}$. Using (1) it is easy to see that

$$\begin{aligned}
&(\nabla_Y \Theta_a)(X_1, \ldots, X_{n-1}) \\
&\quad = \det(Y, X_1, \ldots, X_{n-1}, a) \\
&\quad\quad - \sum_{i=1}^{n-1} \langle AX_i, Y\rangle \det(x, X_1, \ldots, X_{i-1}, N, X_{i+1}, \ldots, X_{n-1}, a)
\end{aligned}$$

for $Y \in \mathcal{X}(M)$. Therefore, the exterior derivative of Θ_a is given by

$$\begin{aligned}
\mathrm{d}\Theta_a(X_1, \ldots, X_n) &= \sum_{i=1}^{n} (-1)^i (\nabla_{X_i}\Theta_a)(X_1, \ldots, \hat{X}_i, \ldots, X_n) \\
&= n \det(X_1, \ldots, X_n, a).
\end{aligned}$$

In other words,

$$\mathrm{d}\Theta_a = -n\langle a, N\rangle \,\mathrm{d}V. \tag{6}$$

When the mean curvature H is constant, Eq. (6) allows us to express the integral on the right-hand side of (5) as an integral over the boundary. This yields a flux formula for immersed spacelike hypersurfaces with constant mean curvature in the Lorentz–Minkowski space. The corresponding formula for hypersurfaces in Euclidean space can be found in [14] (see also [8,12] for the case of hypersurfaces in hyperbolic space).

Lemma 1 (Flux formula). *Let $x : M \longrightarrow \mathbf{L}^{n+1}$ be a spacelike immersion of a compact hypersurface with boundary ∂M. If the mean curvature H is constant, then for any fixed vector $a \in \mathbf{L}^{n+1}$ we have*

$$\oint_{\partial M} \langle \nu, a \rangle \, \mathrm{d}S = -H \oint_{\partial M} \Theta_a,$$

where ν is the outward pointing unit conormal vector along ∂M and

$$\Theta_a(v_1, \ldots, v_{n-1}) = \det(x, v_1, \ldots, v_{n-1}, a)$$

for $v_1, \ldots, v_{n-1}$ tangent to ∂M.

Let us assume from now on that the boundary $\Sigma = x(\partial M)$ is contained in a fixed hyperplane Π of $\mathbf{L}^{n+1}$. Since Σ is closed, it follows that the hyperplane Π is spacelike. We can assume without loss of generality that Π passes through the origin and $\Pi = a^{\perp}$, for a unit future-directed timelike vector $a \in \mathbf{L}^{n+1}$. As a first application of the flux formula we have the following result.

Proposition 2. *Let $x : M \longrightarrow \mathbf{L}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that Σ is contained in a hyperplane Π of $\mathbf{L}^{n+1}$. Let a be the unit future-directed timelike vector in $\mathbf{L}^{n+1}$ such that $\Pi = a^{\perp}$. If the mean curvature H is constant, then the flux*

$$\oint_{\partial M} \langle \nu, a \rangle \, \mathrm{d}S$$

does not depend on the hypersurface, but only on the value of H and Σ. Actually,

$$\oint_{\partial M} \langle \nu, a \rangle \, \mathrm{d}S = -nH \operatorname{vol}(\Omega), \tag{7}$$

where Ω is the domain in Π bounded by Σ.

Proof. From Lemma 1 it suffices to show that

$$\oint_{\partial M} \Theta_a = n \operatorname{vol}(\Omega).$$

Since $\Sigma = x(\partial M)$ is an embedded closed hypersurface in the spacelike hyperplane Π, a well-known (Euclidean) formula gives

$$\operatorname{vol}(\Omega) = \frac{1}{n} \oint_{\partial M} \langle x, \eta \rangle \, \mathrm{d}S,$$

where η is the outward pointing unitary normal to Σ in Π. Here, we are considering on Π the natural orientation determined by a. Observe that if $\{e_1, \ldots, e_{n-1}\}$ is a (locally defined) positively oriented tangent orthonormal frame along ∂M, then $\{\eta, e_1, \ldots, e_{n-1}, a\}$ is positively oriented and $\det(\eta, e_1, \ldots, e_{n-1}, a) = 1$. This implies that

$$\Theta_a(e_1, \ldots, e_{n-1}) = \langle x, \eta \rangle,$$

so that

$$\oint_{\partial M} \Theta_a = \oint_{\partial M} \langle x, \eta \rangle \,\mathrm{d}S = n \,\mathrm{vol}(\Omega),$$

which finishes the proof. □

It is interesting to remark that, in contrast to the Euclidean case, Eq. (7) does not imply here any restriction on the possible values of the constant mean curvature. For example, if $\Sigma = \mathbf{S}^{n-1}$ is an $(n-1)$-dimensional sphere of radius one and M is an immersed compact hypersurface in the Euclidean space bounded by $\mathbf{S}^{n-1}$ with constant mean curvature H, then the corresponding flux formula implies that $0 \le |H| \le 1$ (see [3,4]). However, in the case of the Lorentz–Minkowski space, the family of hyperbolic caps

$$M_\lambda = \{x \in \mathbf{L}^{n+1} : \langle x, x \rangle = -\lambda^2, 0 < x_{n+1} \le \sqrt{1+\lambda^2}\},$$

with $0 < \lambda < \infty$, describes a family of spacelike compact hypersurfaces in $\mathbf{L}^{n+1}$ bounded by $\mathbf{S}^{n-1}$ with constant mean curvature $H_\lambda = 1/\lambda$.

4. An integral inequality

In this section we will derive an integral inequality which, jointly with Proposition 2, will yield our main result. Given a fixed arbitrary vector $a \in \mathbf{L}^{n+1}$, let us consider now the function $\langle a, N \rangle$ defined on M, whose gradient is

$$\nabla \langle a, N \rangle = -A(a^{\mathrm{T}}).$$

From here, and using (1) and (2), it can be seen that the Hessian of $\langle a, N \rangle$ is given by

$$\nabla^2 \langle a, N \rangle (X, Y) = -\langle (\nabla A)(a^{\mathrm{T}}, X), Y \rangle + \langle a, N \rangle \langle AX, AY \rangle,$$

for $X, Y \in \mathcal{X}(M)$. Using now the Codazzi equation,

$$(\nabla A)(a^{\mathrm{T}}, X) = (\nabla A)(X, a^{\mathrm{T}}),$$

it follows that the Laplacian of $\langle a, N \rangle$ is

$$\Delta \langle a, N \rangle = -\mathrm{tr}(\nabla_{a^{\mathrm{T}}} A) + \langle a, N \rangle \mathrm{tr}(A^2) = n \langle \nabla H, a \rangle + \langle a, N \rangle \mathrm{tr}(A^2). \tag{8}$$

When H is constant, from (4) and (8), we obtain that

$$\Delta (H \langle a, x \rangle - \langle a, N \rangle) = -\langle a, N \rangle (\mathrm{tr}(A^2) - nH^2). \tag{9}$$

This equation is the key for the proof of the following result.

Proposition 3. *Let $x : M \longrightarrow \mathbf{L}^{n+1}$ be a spacelike immersion of a compact hypersurface bounded by an $(n-1)$-dimensional embedded submanifold $\Sigma = x(\partial M)$, and assume that*

Σ is contained in a hyperplane Π of $\mathbf{L}^{n+1}$. Let a be the unit future-directed timelike vector in $\mathbf{L}^{n+1}$ such that $\Pi = a^{\perp}$. If the mean curvature H is constant, then

$$-\oint_{\partial M} H_{\Sigma}\langle \nu, a\rangle^2 \,\mathrm{d}S \leq nH^2 \,\mathrm{vol}(\Omega), \tag{10}$$

where Ω is the domain in Π bounded by Σ and H_{Σ} stands for the mean curvature of Σ in Π with respect to the outward pointing unitary normal η. Moreover, the equality holds if and only if M is a totally umbilical hypersurface in $\mathbf{L}^{n+1}$.

Proof. Integrating (9) on M we obtain that

$$\oint_{\partial M} (\langle A(\nu), a\rangle + H\langle \nu, a\rangle)\,\mathrm{d}S = -\int_M \langle a, N\rangle(\mathrm{tr}(A^2) - nH^2)\,\mathrm{d}V.$$

Since N and a are both unit future-directed timelike vectors, then $\langle a, N\rangle \leq -1 < 0$. Moreover, it follows from the Schwarz inequality that $\mathrm{tr}(A^2) - nH^2 \geq 0$, and the equality holds if and only if M is a totally umbilical hypersurface. Therefore,

$$\oint_{\partial M} (\langle A(\nu), a\rangle + H\langle \nu, a\rangle)\,\mathrm{d}S \geq 0, \tag{11}$$

with equality if and only the hypersurface is totally umbilical. Let $\{e_1, \ldots, e_{n-1}\}$ be a (locally defined) positively oriented tangent orthonormal frame along ∂M. Since $\langle a, x\rangle = 0$ on the boundary, then $\langle a, e_i\rangle = 0$ for every $1 \leq i \leq n-1$, and $a^{\mathrm{T}} = \langle a, \nu\rangle\nu$, so that

$$\langle A(\nu), a\rangle = \langle \nu, a\rangle\langle A(\nu), \nu\rangle = \langle \nu, a\rangle \left(\mathrm{tr}(A) - \sum_{i=1}^{n-1}\langle A(e_i), e_i\rangle\right). \tag{12}$$

Let A_{Σ} denote the shape operator of Σ with respect to η. Then, for every $1 \leq i \leq n-1$ we have

$$\nabla^{\mathrm{o}}_{e_i} e_i = \sum_{j\neq i}\langle \nabla^{\mathrm{o}}_{e_i} e_i, e_j\rangle e_j + \langle \nabla^{\mathrm{o}}_{e_i} e_i, \nu\rangle\nu - \langle A(e_i), e_i\rangle N,$$

and also

$$\nabla^{\mathrm{o}}_{e_i} e_i = \sum_{j\neq i}\langle \nabla^{\mathrm{o}}_{e_i} e_i, e_j\rangle e_j + \langle A_{\Sigma}(e_i), e_i\rangle \eta,$$

so that

$$\langle A(e_i), e_i\rangle = \langle A_{\Sigma}(e_i), e_i\rangle\langle \eta, N\rangle = -\langle A_{\Sigma}(e_i), e_i\rangle\langle \nu, a\rangle,$$

since $\langle \eta, N\rangle = -\langle \nu, a\rangle$. Using this in (12) we get

$$\begin{aligned}\langle A(\nu), a\rangle + H\langle \nu, a\rangle &= -(n-1)H\langle \nu, a\rangle + \langle \nu, a\rangle^2 \mathrm{tr}(A_{\Sigma}) \\ &= -(n-1)H\langle \nu, a\rangle + (n-1)H_{\Sigma}\langle \nu, a\rangle^2.\end{aligned}$$

From here and using Proposition 2, (11) becomes

$$\oint_{\partial M} H_\Sigma \langle \nu, a\rangle^2 \, \mathrm{d}S \geq H \oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}S = -nH^2 \operatorname{vol}(\Omega).$$

5. Proof of the theorem

When the boundary $\Sigma = x(\partial M)$ is a round sphere $\mathbf{S}^{n-1}(r)$ of radius $r > 0$, Propositions 2 and 3 imply our uniqueness result. In that case $H_\Sigma = -1/r$, and inequality (10) says that

$$\oint_{\partial M} \langle \nu, a\rangle^2 \mathrm{d}S \leq nH^2 r \operatorname{vol}(\mathbf{B}^n(r)) = H^2 r^2 \operatorname{area}(\mathbf{S}^{n-1}(r)), \tag{13}$$

with equality if and only if M is totally umbilical. On the other hand, by Proposition 2 we also know that

$$\oint_{\partial M} \langle \nu, a\rangle \, \mathrm{d}S = -nH \operatorname{vol}(\mathbf{B}^n(r)) = -Hr \operatorname{area}(\mathbf{S}^{n-1}(r)),$$

and the Cauchy–Schwarz inequality yields

$$\oint_{\partial M} \langle \nu, a\rangle^2 \, \mathrm{d}S \geq H^2 r^2 \operatorname{area}(\mathbf{S}^{n-1}(r)).$$

Therefore, we have the equality in (13) and the result.

Acknowledgements

The first author was partially supported by DGICYT grant no PB94-0750-C02-02 and Consejería de Educación y Cultura CARM Grant No. PB/5/FS/97, Programa Séneca (PRIDTYC) while the second author was supported by an FPPI Predoctoral Grant, DGICYT, 1997.

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